Remark. Similarly, for $m > 2$ and $0 < q, p < m$, we have

$$\left(\sum_{k=1}^{n} \frac{F_k^{m-p}}{L_k^{m-q}}\right) \cdot \left(\sum_{k=1}^{n} \frac{F_k^{p}}{L_k^{q-2}}\right) \geq \frac{(F_{n+2}-1)^m}{(L_{n+2}-3)^{m-2}}.$$

with the same proof.

Also solved by G. C. Greubel, Newport News, VA, USA; Ángel Plaza, University of De Las Palmas, Grain Canaria, Spain; and the proposer.

86.*Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj, Romania.* Calculate

$$\int_0^1 \frac{\ln(\sqrt{x}+\sqrt{1-x})}{\sqrt{x}}dx.$$

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy. Let $x = t^2$. The integral reads as

$$\int_0^1 2\ln(t+\sqrt{1-t^2})dt = 2\int_0^1 \ln t dt + 2\int_0^1 \ln\left(1+\sqrt{\frac{1}{t^2}-1}\right)dt$$

$$\int_0^1 \ln t\, dt = (t\ln t - t)\Big|_0^1 = -1$$

In the second integral we change $t = 1/\sqrt{1+y^2}$ and it becomes

$$\int_0^\infty \ln(1+y)\frac{y}{(1+y^2)^{3/2}}dy = -\frac{\ln(1+y)}{\sqrt{1+y^2}}\Big|_0^\infty + \int_0^\infty \frac{1}{1+y}\frac{1}{\sqrt{1+y^2}}dy$$

$$\underbrace{=}_{y=\sinh t}\int_0^\infty \frac{dt}{1+\sinh t}\underbrace{=}_{z=e^t} 2\int_1^\infty \frac{1}{z^2+2z-1}dz$$

$$\int_1^\infty \left(\frac{1}{z-z_1}-\frac{1}{z-z_2}\right)\frac{1}{z_1-z_2}dz$$

where $z_1 = (-1+\sqrt{2})/2$, $z_2 = (-1-\sqrt{2})/2$. We get evidently

$$\frac{-1}{\sqrt{2}}\ln\frac{1-z_1}{1-z_2} = \frac{-1}{\sqrt{2}}\ln\frac{2-\sqrt{2}}{2+\sqrt{2}} = \frac{-1}{\sqrt{2}}\ln(3-2\sqrt{2}) = \frac{\ln(3+2\sqrt{2})}{\sqrt{2}}$$

and the integral finally is

$$\sqrt{2}\ln(3+2\sqrt{2}) - 2.$$

Solution 2 by Arkady Alt, San Jose, California, USA.

Let $I := \int_0^1 \frac{\ln\left(\sqrt{x}+\sqrt{1-x}\right)}{\sqrt{x}}dx$. The change of variables $x = \sin^2 t$ shows that

$$\begin{aligned} I &= 2\int_0^{\pi/2} \ln(\sin t + \cos t)\cos t\, dt. \\ &= 2\int_0^{\pi/2} \ln(\sin t + \cos t)\sin t\, dt. \qquad \left(t \leftarrow \frac{\pi}{2} - t\right) \end{aligned}$$

Taking the half sum we obtain

$$\begin{aligned}
I &= \int_0^{\pi/2} \ln(\sin t + \cos t)\,(\cos t + \sin t)dt. \\
&= \sqrt{2}\int_{-\pi/4}^{\pi/4} \ln\left(\sqrt{2}\cos\theta\right)\cos\theta\,d\theta \qquad (t \leftarrow \tfrac{\pi}{4} + \theta) \\
&= \frac{\sqrt{2}}{2}(\ln 2)\underbrace{\int_{-\pi/4}^{\pi/4}\cos\theta\,d\theta}_{I_1} + \sqrt{2}\underbrace{\int_{-\pi/4}^{\pi/4}\ln(\cos\theta)\,\cos\theta\,d\theta}_{I_2}
\end{aligned}$$

Clearly, $I_1 = \sqrt{2}$, and

$$\begin{aligned}
I_2 &= \big[\sin\theta\,\ln(\cos\theta)\big]_{-\pi/4}^{\pi/4} + \int_{-\pi/4}^{\pi/4}\frac{\sin^2\theta}{\cos\theta}\,d\theta \\
&= \sqrt{2}\ln\frac{1}{\sqrt{2}} + 2\int_0^{1/\sqrt{2}}\frac{u^2}{1-u^2}du \qquad (u = \sin\theta) \\
&= -\frac{\sqrt{2}}{2}\ln 2 - \sqrt{2} + \int_0^{1/\sqrt{2}}\left(\frac{1}{1+u} + \frac{1}{1-u}\right)du \\
&= -\frac{\sqrt{2}}{2}\ln 2 - \sqrt{2} + \ln\frac{\sqrt{2}+1}{\sqrt{2}-1} \\
&= -\frac{\sqrt{2}}{2}\ln 2 - \sqrt{2} + 2\ln(\sqrt{2}+1)
\end{aligned}$$

Finally $I = -2 + 2\sqrt{2}\ln(\sqrt{2}+1)$.

Also solved by Albert Stadler, Switzerland; Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; Anastasios Kotronis, Athens, Greece; G. C. Greubel, Newport News, VA, USA; Moti Levy, Rehovot, Israel; Moubinool Omarjee, Paris, France, AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; and the proposer.

87.*Proposed by Dorlir Ahmeti, University of Prishtina, Republic of Kosova.* Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{\sqrt{a}+\sqrt{b}}{1+\sqrt{ab}} + \frac{\sqrt{b}+\sqrt{c}}{1+\sqrt{bc}} + \frac{\sqrt{c}+\sqrt{a}}{1+\sqrt{ca}} \geq 3.$$

Solution 1 by AN-anduud Problem Solving Group. The proposed problem is equivalent to the following problem. x, y, z be positive real number such that $x^2 + y^2 + z^2 = 3$. Prove that

$$\frac{x+y}{1+xy} + \frac{y+z}{1+yz} + \frac{z+x}{1+zx} \geq 3. \qquad (1)$$

Applying Hölder's inequality, we have

$$\begin{aligned}
&\left(\frac{x+y}{1+xy} + \frac{y+z}{1+yz} + \frac{z+x}{1+zx}\right)^2((x+y)(1+xy)^2 + (y+z)(1+yz)^2 + (z+x)(1+zx)^2) \\
&\qquad \geq ((x+y) + (y+z) + (z+x))^3 = 8(x+y+z)^3. \qquad (2)
\end{aligned}$$